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OPTIMIZATION PROBLEM OF FOREIGN RESERVES

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Abstract. In this work, we propose non-portfolio optimization problem with one stochastic factor in continuous time. Based on Merton's investment model [9], we construct a mathematical model for foreign reserves of Mongolia. We reduce it to econometric model and estimate statistical parameters.

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1. Introduction

There is a lot of research papers on strategic asset allocations but less attention has been paid foreign reserves allocation. Particularly, the notion "strategic asset allocation" was introduced in Brennan et al. [3] to describe the portfolio optimization problem with time-varying returns and long-term investor objectives. In general, the problem of long-term investments is a well-established research field introduced by Samuelson [11] and Merton [8], respectively.

In the continuous time, using stochastic optimal control theory, Merton [8] has developed the problem of lifetime portfolio selection under uncertainty. In [8] important financial economic principles were established, but there are no explicit results for portfolio choice problems. Merton's paper [9] highlights the difficulties in solving complex cases of assets dynamics with stochastic factors.

Advances in numerical techniques and the growth of computing power led to the development of numerical solutions to multiperiod portfolio optimization problems, which are solved by a discrete state approximation [3,4]. However, the use of numerical dynamic programming is very often restricted to few factors, due to the fact that the algorithms use excessive computation time and become numerically unreliable for high dimensions. Therefore, closed-form solutions of the Merton model in continuous time with two or three stochastic factors are given in [5, 6, 10].

In [2], Bielecki et al. present a closed-form solution to the portfolio optimization problem in continuous time for multiple assets and multiple factors with an infinite time horizon. Under the assumption of the uncorrelated residual of the asset prices and factors, they find the optimal portfolio allocation decision for many assets and many factors. In [2], general analytical solutions Merton's type consumption-investment problem were obtained. Brownian motion with a white noise was examined in [2].

Based on [7], we suppose non-portfolio problem in the continuous time with one stochastic factor controls optimal value of foreign reserves when the exchange rates fluctuate.

2. Merton's type consumption-investment problem

In our case the investor lives from 0 to time T; his wealth at time t is denoted byX_t. The investor starts with a known initial wealthX_0. At time $t$the investor must choose what fraction of his wealth to consume c_t and what fraction to invest in the riskless and risky portfolio.

Let's assume that $F(t, c_t)$ is the period utility function for consumption at time t, and $\Phi(T, X_T)$ is the boundary condition. Then the optimization problem for the investor is

$$\max_{u^0, u^1, c} E\left[\int_0^T F(t, c_t)dt + \Phi(T, X_T)\right], \tag{1}$$

where E is the expectation operator, T is the planning horizon.
Suppose there are two assets.

1. The investor can invest money in the bank at the deterministic short rate of interest r, i.e, the investor has access to the risk free asset B with

$$dB(t) = rB(t)dt. \tag{2}$$

The investor can invest in a risky asset with price process S, where we assume that the S-dynamics are given by a standard Black-Scholes model

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW, \tag{3}$$

where α: drift function, σ: diffusion coefficient, $W(t)$: standard Brownian motion.

If we assume that, at time t the investor owns n_1, n_2 units of asset, then the total wealth is determined by

$$X(t) = n_1(t)B(t) + n_2(t)S(t). \tag{4}$$

The portfolio $\{n_1, n_2\}$ remains unchanged over the time interval $[t, t + \Delta t[$, and assuming the consumption pattern is constant in interval $[t,t + \Delta t)$ in the same way as for portfolio selection, the budget equation becomes

$$dX(t) = n_1(t)dB(t) + n_2(t)dS(t) - c(t)dt \text{ (5).}$$

Let

$$u^0(t) = \frac{n_1(t)B(t)}{X(t)},\ u^1(t) = \frac{n_2(t)S(t)}{X(t)}, \tag{6}$$

be the share of wealth in assets, with

$$u^0(t) + u^1(t) = 1. \tag{7}$$

Then the budget equation (5) can be written as

$$dX(t) = X(t)[u^0(t)r + u^1(t)\alpha]dt - c(t)dt + u^1(t)\sigma X(t)dW(t). \tag{8}$$

Now, we may formally state the consumer's utility maximization problem a stochastic optimal control problem

$$\max_{u^0, u^1, c} E\left[\int_0^T F(t, c_t)dt + \Phi(T, X_T)\right],$$
$$dX(t) = X(t)[u^0(t)r + u^1(t)\alpha]dt - c(t)dt + u^1(t)\sigma X(t)dW(t), \tag{9}$$

$$X_0 = x_0,$$
$$c(t) \geq 0, \ \forall t \geq 0,$$
$$u^0(t) + u^1(t) = 1, \qquad \forall t \geq 0,$$

where X is a state process, u^0, u^1 and c are control variables.

3. Foreign reserves problem

In problem (1.9), we assume the following.

1. There is only one stochastic factor described by the risky asset.
2. Social welfare is increasing due to foreign reserves.
3. Changes of foreign reserves is not constant.

There are $u^0(t) = 0, u^1(t) = w(t)$ for (1), $F(t, c_t) = e^{-\delta t} R_t^\gamma$ for (2), $c(t) = \beta R(t)$ for (3), and $\Phi(T, X_T) = 0$ from (4).

According to assumptions, (1.9) can be written as follows

$$\max_{w,R} E\left[\int_0^T e^{-\delta t} R_t^\gamma \, dt\right],$$

$$dX(t) = w(t)\alpha X(t)dt - \beta R(t)dt + w(t)\sigma X(t)dW(t), \tag{10}$$
$$X_0 = x_0,$$
$$R(t) \geq 0, \qquad \forall t \geq 0,$$

$$0 \leq w(t) \leq 1, \qquad \forall t \geq 0.$$

The Hamilton-Jacobi-Bellman (HJB) equation has the following form

$$\frac{\partial V}{\partial t} + sup_{R \geq 0, w \in \mathbb{R}} \left\{ e^{-\delta t} R_t^\gamma + w\alpha X \frac{\partial V}{\partial X} - \beta R \frac{\partial V}{\partial X} + \frac{1}{2} w^2 \sigma^2 X^2 \frac{\partial^2 V}{\partial X^2} \right\} = 0. \tag{11}$$

Let R and w be solution problem (11). Assuming an interior solution, the first order conditions are

$$\gamma R^{\gamma-1} = \beta e^{\delta t} V_X, \tag{12}$$

$$w = -\frac{\alpha V_X}{X\sigma^2 V_{XX}}. \tag{13}$$

Taking into account assumption (2), we have

$$V(t, X) = e^{-\delta t} h X^\gamma.$$

The boundary conditions require that

$$\Phi(T, X_T) = 0.$$

Also, we can get the following equations

$$\frac{\partial V}{\partial t} = e^{-\delta t} \dot{h} X^\gamma - \delta e^{-\delta t} h X^\gamma, \tag{14}$$

$$\frac{\partial V}{\partial X} = \gamma e^{-\delta t} h X^{\gamma-1}, \tag{15}$$

$$\frac{\partial^2 V}{\partial X^2} = \gamma(\gamma-1)e^{-\delta t} h X^{\gamma-2}. \tag{16}$$

Substituting (15) into (12), (16) and (13), we get

$$R^*(t,X) = \left(\beta h(t)\right)^{-1/(1-\gamma)} X, \tag{17}$$

$$w^*(t,X) = \frac{\alpha}{(1-\gamma)\sigma^2}. \tag{18}$$

If we substitute the expressions (14) and (17)-(18) into the HJB equation, we obtain:

$$X^\gamma \left[\dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)}\right] = 0, \tag{19}$$

where A and B are the constants given by

$$A = \frac{1}{2}\frac{\alpha^2\gamma}{(1-\gamma)\sigma^2} - \delta,$$

$$B = (1-\gamma)\beta^{-\gamma/(1-\gamma)}.$$

If we search $y$in the following form$y = h^{1/(1-\gamma)}$, we get

$$y(t) = \left(y(0) - \frac{B}{A}\right)e^{-\frac{A}{1-\gamma}t} + \frac{B}{A}. \tag{20}$$

We can easily see that

$$\lim_{t\to\infty} y(t) = \frac{B}{A}.$$

Now we can find the optimal value of foreign reserves using the expression (20)

$$R^*(t,X) = \beta^{-1/(1-\gamma)} X \frac{A}{B}. \tag{21}$$

From here we find expected foreign reserves:

$$E(R^*) = \beta^{-1/(1-\gamma)} \frac{A}{B} E(X). \tag{22}$$

4. Estimation of econometric model

From (1.10), a stochastic differential equation is written as

$$\frac{dX(t)}{X(t)} = w(t)\alpha dt - \beta\frac{R(t)}{X(t)}dt + w(t)\sigma dW(t).$$

In order to construct econometric model, assume that $w(t) = w$ and $w\sigma dW(t) = u(t)$. Then we have the following econometric equation:

$$dln(X_t) = w\alpha - \beta\frac{R_t}{X_t} + u_t.$$

First-order autocorrelation is:

$$u_t = \rho u_{t-1} + e_t.$$

where ρ can be computed

$$\rho = \frac{Cov\left(w\sigma dW(t), w\sigma dW(s)\right)}{Var\left(w\sigma dW(t)\right)}. \tag{23}$$

Corollary 1. If s, t are positive, then

$Cov\big(dW(t), dW(s)\big) = Var\big(dW(t)\big)\delta(s-t)$ for the differential process, where $\delta(s-t)$ is continuous Dirac delta function. If we set δ as $\delta(s-t)=1$, then ρ becomes

$$\rho = \frac{w\sigma}{w^2\sigma^2}\frac{Cov\big(dW(t), dW(s)\big)}{Var\big(dW(t)\big)} = \frac{1}{w\sigma}\frac{Var\big(dW(t)\big)\delta(s-t)}{Var\big(dW(t)\big)} = \frac{1}{w\sigma}.$$

Corollary 2. Let $R(t)$ be a foreign reserves with an exponential growth $R(t) = R(0)e^{\frac{\tau}{\gamma}t}$. If objective function is

$$\max_{w,R} E\left[\int_0^T e^{-\delta t} R_t^{\gamma}\, dt\right] > 0\,.$$

Then, there exists τ such $\tau > \delta$.
Proof follows from

$$\max_{w,R} E\left[\int_0^R e^{-r_i t} R^{\gamma}(0) e^{\tau t}\, dt\right] = \frac{R^{\gamma}(0)}{\tau-\delta}\left[e^{(\tau-\delta)T} - 1\right] > 0\,.$$

If $\tau - \delta > 0$, the objective function is positive which proves the assertion.
Changes in (1.18) can be written:

$$w^* = \frac{(\alpha w)w}{(1-\gamma)(\sigma w)^2}\,.$$

From the above, we have

$$\gamma = 1 - \frac{\alpha w}{(\sigma w)^2} = 1 - \alpha w \rho^2\,.$$

Optimal level of foreign reserves is

$$E(R^*) = \beta^{-\frac{1}{(1-\gamma)}}\frac{A}{B}E(X) = \beta^{-\frac{1}{(1-\gamma)}}\frac{\frac{1}{2}\frac{\alpha^2\gamma}{(1-\gamma)\sigma^2} - \delta}{(1-\gamma)\beta^{-\frac{\gamma}{(1-\gamma)}}}E(X) =$$

$$= \frac{1}{\beta(1-\gamma)}\left(\frac{1}{2}\left[\frac{\alpha}{(1-\gamma)\sigma^2}\right]\alpha\gamma - \delta\right)E(X) = \frac{1}{\beta(1-\gamma)}\left(\frac{1}{2}\alpha w\gamma - \tau\right)E(X)\,.$$

5. Computational results

For our econometric model, we used observations of foreign reserves and exchange rate for 132 months during 2006-2016. The first-order autocorrelation estimation gives the following values:

$$\frac{dX(t)}{X(t)} = 0.182 - 0.072 \cdot \frac{R(t)}{X(t)} + 0.957 \cdot AR(1)\ .$$

If we look at estimation results, all parameters of the equation are statistically significant.

The quarterly average growth of foreign reserves is

$$\frac{\tau}{\gamma} = 0.066\,.$$

Also, expected exchange rate level is

$$E(X) = 1465.715\,.$$

Thus, we final all parameters as follows

$$w\alpha = 0.182,$$

$$\beta = 0.072,$$

$$\rho = 0.957.$$

Dependent Variable: (USD-USD(-12))/USD(-12)
Method: Least Squares
Sample (adjusted): 2006M01 2016M12
Included observations: 132 after adjustments
Convergence achieved after 8 iterations

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.182294	0.075582	2.411877	0.0173
RES/USD	-0.071848	0.022383	-3.209863	0.0017
AR(1)	0.956635	0.026770	35.73492	0.0000

R-squared	0.908303	Mean dependent var	0.059319
Adjusted R-squared	0.906870	S.D. dependent var	0.108805
S.E. of regression	0.033204	Akaike info criterion	-3.949639
Sum squared resid	0.141123	Schwarz criterion	-3.883795
Log likelihood	261.7014	Hannan-Quinn criteria.	-3.922884
F-statistic	633.9524	Durbin-Watson stat	1.459133
Prob(F-statistic)	0.000000		

Inverted AR Roots	.96

If we compute then

$$\gamma = 1 - 0.182 \cdot 0.957^2 = 0.8333,$$
$$\tau = 0.8333 \cdot 0.066 = 0.055$$

and expected the optimal value of foreign reserves is obtained by 2.5 billion USD.

$$E(R^*) = \frac{1}{\beta(1-\gamma)}\left(\frac{1}{2}\alpha w\gamma - \tau\right)E(X) =$$
$$= \frac{1465.715}{0.072\cdot(1-0.8333)} \cdot \left(\frac{1}{2} \cdot 0.182 \cdot 0.8333 - 0.055\right) \approx 2544.$$

6. Conclusion

We solved non-portfolio optimization problem with one stochastic factor in continuous time. This problem reduced to an econometric model with significant statistical parameters. Using this result, we evaluated optimal value of foreign reserves of Mongolia.

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